Journal of Thermal Analysis, Vol. 20 (1981) 369—378

EXPONENTIAL-LIKE RESPONSES ANALYSIS METHOD

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(Received June 19, 1980)

An exponential-like responses analysis method is presented. The method enables one: 1) to evaluate the similarity of a curve to the ideal exponential characteristic; 2) to find the most appropriate final value of a curve; and 3) to find the value of τ giving the best approximation between the analyzed curve and the ideal exponential one.

Various physical processes (mainly transient) are governed by the exponential law:

$$\Delta A_{\text{trans.}}(t) = \left[1 - \exp\left(\frac{t}{\tau}\right)\right] \Delta A_{\text{fin.}} \tag{1}$$

where: $\Delta A_{\text{trans.}}(t)$ is the transient value of an increment of a magnitude A, changing as a function of t;

$\Delta A_{\text{fin.}}$ is the final value of an increment of a magnitude A, usually caused by a step change of a certain parameter influencing A; it means that $\Delta A_{\text{fin.}} = A(t \to \infty) - A(t = 0) = \Delta A_{\text{trans.}}(t \to \to \infty)$;

t is an independent variable whose function is $\Delta A_{\text{trans.}}$ (usually time);

τ is a parameter which exists in a function $\Delta A_{\text{trans.}}(t)$ (usually a time constant).

In a general case A may be any magnitude such as voltage, frequency, length, velocity, electrical resistance, and so on. Similarly, in a general case the independent variable t may have another meaning than time. An example of a characteristic represented by Eq. (1) is given in Fig. 1. It should be noted that on the $\Delta A_{\text{trans.}}$ axis various above-mentioned magnitudes may appear (velocity, length, resistance, and so on) and the value of $\Delta A_{\text{trans.}}$ (and $\Delta A_{\text{fin.}}$) may be expressed by any number.

Let us transform Eq. (1) into another form, which is more convenient for analytical purposes:

$$F_1 = \frac{\Delta A_{\text{trans.}}(t)}{\Delta A_{\text{fin.}}} = 1 - \exp\left(-\frac{t}{\tau}\right). \tag{2}$$

A graphical illustration of this equation is presented in Fig. 2a. The diagram has a similar shape as the curve of Fig. 1, but it is now normalized, that is, a magnitude represented on the y-axis is dimensionless and the curve always goes to a saturation level equal to 1.

Processes described by Eqs (1) and (2) and illustrated in Fig. 1 and Fig. 2a are completely determined by $\Delta A_{\mathrm{fin.}}$ and τ [2]. In a theoretical analysis the factor $\Delta A_{\mathrm{fin.}}$ is usually defined from a function describing the dependence between a

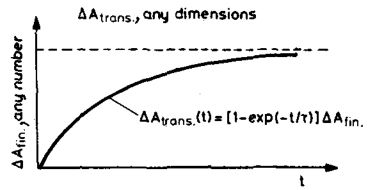

Fig. 1. Illustration of a typical transient process

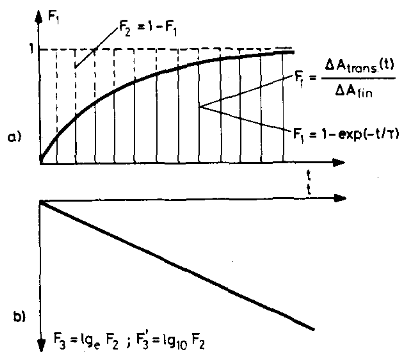

Fig. 2. Diagrams: a) of an exponential function F_1; b) of its transformation defined by means of function F_3

magnitude A and this parameter, which causes its increment, and the parameter τ is usually calculated from appropriate analytical expressions; for instance, in electrical circuits $\Delta A_{\mathrm{fin.}}$ is a function of the change of a current or of a voltage and τ is a function of RLC values. Thus, in a theoretical analysis components of Eqs (1) and (2) are given by appropriate analytical formule and both (Fig. 1 and Fig. 2a) characteristics may be precisely calculated point by point.

An opposite situation occurs in experimental research work. A characteristic is given and one has to determine its analytical parameters. In the simplest case one knows or assumes that an experimentally obtained curve has a perfect exponential form, and in Eqs (1) or (2) one wants to determine:

a) the parameter τ, assuming that an obtained $\Delta A_{\mathrm{fin.}}$ value is true;

b) the magnitude $\Delta A_{fin.}$, assuming that the parameter τ is known. In particular, it may concern cases where an investigated process is very slow and there is a need to determine $\Delta A_{fin.}$ on the basis of the initial part of the characteristic.

In these two cases both parameter τ and $\Delta A_{fin.}$ may easily be calculated for any point of a curve, using formulae which are derived directly from Eqs (1) or (2): Case *a* ($\Delta A_{fin.}$ is known):

$$\tau = -\frac{t_i}{\log_e\left[1 - \left[\frac{\Delta A_{trans.}(t_i)}{\Delta A_{fin.}}\right]\right]} = -\frac{0.4343\, t_i}{\log_{10}\left[1 - \frac{\Delta A_{trans.}(t_i)}{\Delta A_{fin.}}\right]} \tag{3}$$

Case *b* (τ is known):

$$\Delta A_{fin.} = \frac{\Delta A_{trans.}(t_i)}{1 - \exp\left(-\frac{t_i}{\tau}\right)} \tag{4}$$

where t_i is any arbitrarily chosen point in the t-axis. Of course, a common practice is to calculate several τ or $\Delta A_{fin.}$ values for several t_i points of the curve, and to take their average value as a final result.

As has been mentioned above, the simplest case is when there are no doubts concerning either the exponential character of a curve, or one of the parameters which is actually given, e.g. $\Delta A_{fin.}$ or τ. Usually there are some uncertainties about the character of the curve itself, as well as about the given parameter $\Delta A_{.fin.}$ or τ. These uncertainties results from various reasons, the most common of them being the following:

– in experimental investigations it happens that one does not know what analytical law governs an obtained characteristic, so one may not be sure if the curve is really an exponential one, and this has to be checked;

– even characteristics which should theoretically be exponential may have some deformations, caused partially by the measurement system and partially by certain imperfections in the experimental conditions (for example in the thermal processes it may be grad $T \neq 0$ or a thermal step having a fairly long time of change between two temperatures);

– in the case of fairly slow processes one does not have reliable information about the correct final value ΔA_{fin} of the curve, because an experiment has not proceeded for a long enough time or the sensitivity of the measuring system is too low for small increments in the magnitude of A which take place in the final part of an experiment (in such cases the $\Delta A_{fin.}$ value has to be determined by extrapolation and it is necessary to check if this value has been chosen correctly);

– even when the parameter τ is given as a known value (for example calculated from an analytical expression) one should treat it critically because its magnitude, as a function of certain parameters (measured of course with some error), may be determined only with a limited degree of accuracy.

It is clear from the above remarks that a full analysis of a curve should enable one: 1) to evaluate the similarly of a curve to the ideal exponential characteristic; 2) to find the most appropriate final value of a curve, $\Delta A_{\text{fin.}}$; 3) to find the value of giving the best approximation between the analyzed curve and the ideal exponential one. As will be shown below, the analysis method presented here meets these requirements to a considerable degree.

The basic idea of the method is generally well known and has been used fairly widely to solve various problems. It consists in the analysis of certain experimental data by looking for some mathematical transformation which can be applied to the experimental data so that these data will result in a straight line if they fit a certain type of function. If one wants to apply such a procedure to Eq. (2), it appears that it is rather difficult to find a transformation which enables one to present function (2) as a straight line, or that transformations which may exist are very inconvenient for practical use. However, it appears that a convenient solution does exist, which consists not in analyzing a function of the form expressed by Eq. (2), but in analyzing the difference between unity and this function:

Case a ($\Delta A_{\text{fin.}}$ is known):

$$F_2 = 1 - \frac{\Delta A_{\text{trans.}}(t)}{\Delta A_{\text{fin.}}} . \tag{5}$$

Case b (τ is known):

$$F_2 = \exp\left(-\frac{t}{\tau}\right) . \tag{6}$$

The above equations are obtained by a trivial transformation of Eq. (2). The relations expressed by these two equations may also be observed directly in the typical diagrams illustrated in Fig. 2a where it is shown that the same curve may represent simultaneously function (2) (continuous ordinates) and function (5) or (6)* (dotted ordinates).

Analysis of function (2) is now achieved by means of analysis of function (5) or (6)**. Its numerical values are expressed simply by the lengths of the dotted ordinates. Having values of the function (5) or (6) for individual values of the variable t, one calculates values of one of two functions:

Case a (ΔA_{fin}- is known):

$$F_3 = \log_e F_2 = \log_e\left[1 - \frac{\Delta A_{\text{trans.}}(t)}{\Delta A_{\text{fin.}}}\right];\ F_3' = \log_{10} F_2 = \\ = \log_{10}\left[1 - \frac{\Delta A_{\text{trans,}}(t)}{\Delta A_{\text{fin.}}}\right]. \tag{7; 7'}$$

* depending on what is given: $\Delta A_{\text{fin.}}$ or τ.

** depending on what is given: $\Delta A_{\text{fin.}}$ or τ.

Case b (τ is known):

$$F_4 = \log_e F_2 = \log_e \left[\exp\left(-\frac{t}{\tau}\right)\right] = -\frac{t}{\tau}\,;$$
$$F_4' = \log_{10} F_2 = \log_{10}\left[\exp\left(-\frac{t}{\tau}\right)\right] = -0.4343\,\frac{t}{\tau} \qquad (8;\ 8')$$

for these individual values of the variable t.

In case a ($\Delta A_{\mathrm{fin.}}$ is known, τ has to be determined) it is necessary now to draw an $F_3(t)$ or $F'(t)$ diagram (Fig. 2b). When the original characteristic has an expo-

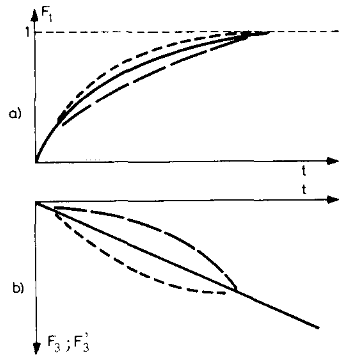

Fig. 3. Deformations of exponential characteristics described by (a) the function F_1 and (b) the corresponding changes in the function F_3

nential character, this diagram is a straight line, or course. This is illustrated in Fig. 2b. If the original curve has some deformations, the function $F_3(t)$ (or $F_3'(t)$) exhibits deviations from a straight line. Examples of typical deformations of exponential-like characteristics and (corresponding to these) the forms of deformations of the diagrams of the functions $F_3(t)$ (or $F_3'(t)$) are shown in Fig. 3 and Fig. 4.

Figure 3 shows the situation when the experimentally obtained characteristics are more or less convex than the ideal exponential characteristic (Fig. 3a, short and long dotted curves, respectively). The corresponding shapes of the $F_3(t)$ (or $F_3'(t)$) function diagrams are presented in Fig. 3b. When the original characteristic (Fig. 3a) is more convex than the ideal exponential characteristic, the corresponding diagram of the function $F_3(t)$ (or $F_3'(t)$) presents a certain concavity. This case is represented by short dotted curves in Fig. 3a and in Fig. 3b. An opposite case (the original response not convex enough) is illustrated by long dotted curves in Fig. 3a and in Fig. 3b. As shown in Fig. 3b, the diagram of the function $F_3'(t)$ (or $F_3'(t)$) then presents a certain convexity.

Figure 4 illustrates the case when the experimentally obtained characteristics have their final values different from 1 (especially in relation to their form in the initial and in the middle parts of the curves). In Fig. 4a a long dotted line represents the situation when an original characteristic has its final value too small in comparison with an ideal exponential curve; a short dotted line shows the opposite case. The corresponding shapes of the diagrams of the function $F_3(t)$ (or $F_3'(t)$) are presented in Fig. 4b. When an original curve has its final value smaller than 1 (in comparison with an ideal exponential characteristic), the corresponding diagram of the function $F_3(t)$ (or $F_3'(t)$) has its final part deviating down (long dotted curve in Fig. 4b). Alternatively, a final value greater than 1 causes an upward deviation of the function $F_3(t)$ (or $F_3'(t)$) (short dotted curve in Fig. 4b).

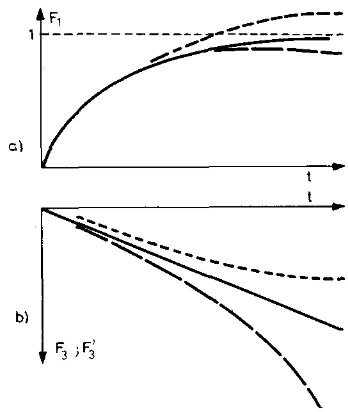

Fig. 4. (a) Exponential characteristics (determined theoretically by the function F_1) having their final values different from unity and (b) the corresponding diagram of the function F_3

When, after a certain averaging or correcting procedure, one has obtained a diagram of the function $F_3(t)$ (or $F_3'(t)$) in the form of a straight line, one may very quickly and easily determine the value of τ from this characteristic.

Case a ($\Delta A_{\text{fin.}}$ is known): As τ is the reciprocal of the slope of the $F_3(t)$ (or $F_3(t)$) diagram and as this line should begin at the origin, it is determined by the following formulae:

$$\tau = -\frac{t_i}{F_3} = \frac{-t_i}{\log_e\left[1 - \frac{\Delta A_{\text{trans.}}(t_i)}{\Delta A_{\text{fin.}}}\right]}; \quad \tau = \frac{-0.4343\, t_i}{F_3'}$$

$$\tau = \frac{-0.4343\, t_i}{\log_{10}\left[1 - \frac{\Delta A_{\text{trans,}}(t_i)}{\Delta A_{\text{fin.}}}\right]} \qquad (9;\ 9')$$

Case b (τ is known): Solving Eqs (8) or (8') or using formula (4) directly, one obtains respectively:

$$\Delta A_{\text{fin.}} = \frac{\Delta A_{\text{trans.}}(t)}{1 - \exp(F_4)} = \frac{\Delta A_{\text{trans.}}(t_i)}{1 - \exp\left(\frac{t_i}{\tau}\right)} \tag{10}$$

$$\Delta A_{\text{fin.}} = \frac{\Delta A_{\text{trans.}}(t)}{1 - 10^{F_4'}} = \frac{\Delta A_{\text{trans.}}(t_i)}{1 - 10^{-0.4343}\,\frac{t_i}{\tau}} \tag{10'}$$

where t_i is any arbitrarily chosen point on the t-axis.

The above method of analysis of exponential-like responses has been found to be very useful in practice. Many responses for a thermal step function of electronic components and arrangements were analyzed in this way by the author [1] and the results obtained showed their correctness and usefulness in further experimental investigations and in mathematical analysis as well. The analysis is applicable of course to any exponential-like curves, and not only to thermal responses. The practical procedure is illustrated in the example presented below.

Example

As an example, use will be made of the thermal response given in the paper [3]. On page 95 of this paper, Fig. 4a shows an oscillogram which represents the experimentally obtained thermal response of the bridgewire (fine wire) of an apparatus. It results from the theoretical analysis presented in the paper that the response should have an exponential character. The response is copied in Fig. 5. It will be analyzed according to the procedure described above. The analysis and its successive steps are presented and explained in Table 1.

Using the numbers obtained in the 5th row, one draws the $F_3'(t)$ diagram, which is shown in Fig. 6. It is seen in Fig. 6 that the diagram obtained is a straight line

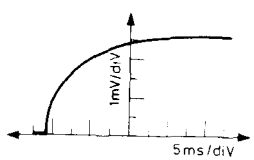

Fig. 5. Oscillogram of the thermal response of a bridgewire

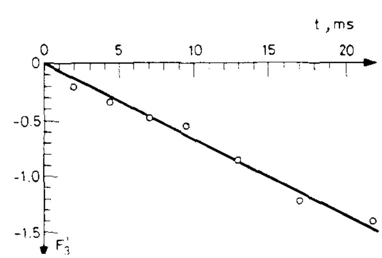

Fig. 6. Diagram of the function $F_3(t)$ for the response illustrated in Fig. 5

Table 1

Analysis of the thermal response of the bridgewire — illustrated in Fig. 5. Determination of τ

Step 1	Choice of points on the t axis	t, ms	2.0	4.5	7.0	9.5	12	17	22	Remarks
2	Reading values of ordinates	$\Delta A_{trans.}$ mV	1.9	2.7	3.3	3.6	4.3	4.7	4.8	Assumed (= read) : $\Delta A_{fin} = 5$ mV
3	Calculating the ratio	$\frac{\Delta A_{trans.}}{\Delta A_{fin.}}$	0.38	0.54	0.66	0.72	0.86	0.94	0.96	
4	Calculating the difference	$1 - \frac{\Delta A_{trans.}}{\Delta A_{fin.}}$	0.62	0.46	0.34	0.28	0.14	0.06	0.04	
5	Calculating the log.	$\log_{10}\left[1 - \frac{\Delta A_{trans.}}{\Delta A_{fin.}}\right]$	−0.20	−0.34	−0.47	−0.55	−0.85	−1.22	−1.4	F_3'

Table 2

Analysis of a thermal response of a bridgewire-illustrated in Fig. 5. Determination of $\Delta A_{fin.}$

Step 1	Choice of points on the t axis	t, ms	2.0	4.5	7.0	9.5	Remarks
2	Reading values of ordinates	$\Delta A_{trans.}$ mV	1.9	2.7	3.3	3.6	
3	Calculating the ratio	$-\frac{t}{\tau}$	−0.322	−0.725	−1.13	−1.53	$\tau = 6{,}2$, ms
4	Calculating the exponen. function	$\exp\left(-\frac{t}{\tau}\right)$	0.724	0.484	0.323	0.216	
5	Calculating the difference	$1 - \exp\left(-\frac{t}{\tau}\right)$	0.276	0.516	0.677	0.784	
6	Calculating the $\Delta A_{fin.}$ values	$\frac{\Delta A_{trans.}}{1 - \exp\left(-\frac{t}{\tau}\right)}$	6.88	5.23	4.87	4.60	Dimension: [mV] formulae: (4) and (10)

(accepting tolerable limits of a certain dispersion of the calculated points). This means that the analyzed characteristic has an exponential character and its final value has been read sufficiently correctly. The time constant τ of this curve is determined using Eq. (9):

$$\tau = \frac{-0.4343 t_i}{F_3'} = \frac{-0.4343\, t_i}{\log_{10}\left[1 - \frac{\Delta A_{\mathrm{trans.}}(t_i)}{\Delta A_{\mathrm{fin.}}}\right]} = \frac{-0.4343.20}{-1.35} 6.2 \text{ [ms]}.$$

If the diagram in Fig. 6 is not a straight line, it would be necessary to take into consideration either a slightly different final value of the curve ($\Delta A_{\mathrm{fin.}}$) or slightly different values of the individual ordinates, depending on the deformations of the $F_3(t)$ (or $F_3'(t)$) diagram. When new values of $\Delta A_{\mathrm{fin.}}$ and $\Delta A_{\mathrm{trans.}}$ are fixed, one has to repeat the set of calculations shown in Table 1 and to draw again the $F_3(t)$ (or $F_3'(t)$) diagram. It may happen of course that the modifications of $\Delta A_{\mathrm{fin.}}$ or $\Delta A_{\mathrm{trans.}}$ necessary to obtain an $F_3(t)$ (or $F_3'(t)$) diagram in the form of a straight line will be unacceptably great. Such a situation means that the analyzed response is considerably different from the exponential one.

Let us assume in turn that only the initial part of the curve illustrated in Fig. 5 is given (namely the part for the first 10 ms), that the time constant is known to be τ = 6.2 ms (according to the calculations in Table 1), and that $\Delta A_{\mathrm{fin.}}$ has to be determined.

The practical procedure for determining $\Delta A_{\mathrm{fin.}}$ is presented in Table 2, where the successive steps of the analysis are also described.

The average value of $\Delta A_{\mathrm{fin.}}$ (calculated on the basis of the obtained 4 values) is $\Delta A_{\mathrm{fin.av}\,(1,2,3,4)}$ = 5.4 mV. If the first value is rejected (possible errors have the highest values at the beginning of the response), the average value is $\Delta A_{\mathrm{fin.av}\,(2,3,4)}$ = = 4.9 mV. The difference between the value read from the response and the calculated value is 8% and 2%, respectively, which seems to be an acceptable degree of accuracy.

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The author wishes to thank Mr. G. B. Hoadley of the IEEE for some remarks which enriched the fundamental idea in the manuscript.

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RÉSUMÉ — On présente une méthode d'analyse des réponses exponentielles. La méthode permet: 1) d'évaluer la similitude d'une courbe à la caractéristique exponentielle idéale, 2) de trouver la valeur finale la plus appropriée d'une courbe, 3) de trouver la valeur τ qui donne la meilleure approximation entre la courbe analysée et l'exponentielle idéale.

ZUSAMMENFASSUNG — Es wird eine Analysenmethode für exponentiell-ähnliche Antworten beschrieben. Die Methode ermöglicht 1) die Ermittlung der Ähnlichkeit einer Kurve mit dem Idealen exponentiellen Charakter, 2) Erforschung des geeigneten Endwertes einer Kurve und 3) Auffindung des Wertes welcher die beste Approximation der analysierten Kurve zur idealen exponentiellen ermöglicht.

Резюме — Представлен метод экспоненциально-подобного анализа отклика. Метод позволяет провести сходство кривой к идеальной экспоненциальной характеристике, выбрать наиболее подходящее конечное значение кривой и найти значение τ, дающее наилучшее приближение между анализированной кривой и идеальной экспонентой.